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A Finite-Time Consensus Framework Over Time-Varying Graph Topologies With Temporal Constraints

Finite-time consensus has attracted significant research interest due to its wide applications in multiagent systems. Various results have been developed to enable multiagent systems to complete desired tasks in finite-time. However, most existing results in the literature can only ensure finite-time consensus without considering temporal constraints, where the time used to achieve consensus cannot be preset arbitrarily and is generally determined by the system initial conditions, prohibiting its application in time-sensitive tasks. Motivated to achieve consensus within a desired time frame, user-specified finite-time consensus is developed in the present work for a multiagent system to ensure consensus at a prespecified time instant. The interaction among agents (e.g., communication and information exchange) is modeled as a time-varying graph, where each edge is associated with a time-varying weight representing the time-varying interaction between neighboring agents. Consensus over such time-varying graph is then proven based on a time transformation and is guaranteed to be completed within a prespecified time frame. To demonstrate the developed framework, finite-time rendezvous of a multiagent system is considered as an example application, where agents with limited communication capabilities are desired to meet at a common location at a preset time instant with constraints on preserving global network connectivity. A numerical simulation is provided to demonstrate the efficiency of the developed result. [DOI: 10.1115/1.4035612]

1 Introduction

Consensus has attracted significant research attention due to its wide applications in rendezvous and flocking problems [1–3], distributed sensing and computation [4–6], and formation control [7–9], to name a few. A comprehensive review of consensus problem is provided in Refs. [10] and [11]. Although numerous results have been developed to achieve consensus, few existing results in the literature consider completing consensus with temporal constraints (i.e., completing consensus within a desired time frame). However, practical applications are generally time-sensitive. For instance, the environment search algorithm developed in Ref. [12] depends on the completion of a consensus-based approximation at each iteration and other applications such as consensus-based rescue, surveillance, and target tracking all demand-ensured consensus within a short period. Therefore, achieving consensus within a desired finite-time frame is a problem of wide interest.

As a particular class of consensus problems, finite-time consensus has received considerable research effort recently. In Ref. [13], finite-time convergence is guaranteed by using the normalized and signed gradient descent flows of a differential function. Finite-time semistability for dynamical systems is introduced and applied for network consensus problems in Ref. [14]. In Ref. [15], consensus problem is investigated for multiagent systems, where only sign information of the relative states between the neighboring agents is used to achieve the finite-time convergence. In Ref. [16], finite-time consensus with respect to a monotonic function is developed for a group of kinematic agents with a time-varying topology to reach the weighted average of their initial values. The finite-time consensus is developed for nonlinear multiagent systems under a general setting of directed and switching topologies in Ref. [17]. Finite-time decentralized formation tracking of single-integrator multiagent systems is investigated in Refs. [18] and [19]. The finite-time formation control is then extended to consider multiagent systems with double-integrator dynamics in Ref. [20] and multiple nonholonomic mobile robots in Ref. [21]. Finite-time consensus is ensured in the aforementioned results of Refs. [13–21]; however, only an upper bound of the convergence time is developed in those results. Such upper bounds typically depend on the initial conditions of the system and cannot be preset arbitrarily, which indicates that the consensus cannot be ensured within an arbitrary-desired time frame.

In practice, completing tasks within a given time frame is paramount in various applications. Example applications include launching munitions from different locations with the objective of simultaneous target arrival at a user-specified time instant or reconfiguring the formation of unmanned aerial vehicles within a given time frame to switch tasks between surveillance and target tracking. It is worth pointing out that the results developed in Refs. [13–21] are not applicable to such applications due to their uncontrollable convergence time. Motivated to achieve consensus at any preset time, linear time-varying feedback control protocols are developed in the works of Refs. [22–25] to achieve consensus, containment control, and circle formation within a desired time frame. However, the results developed in Refs. [22–25] are only limited to time-invariant interaction graphs. Since mission operation within dynamic and complex environments can result in time-varying interaction and complicated coordination between agents, achieving consensus among agents at a preset time instant over such time-varying graphs could be very challenging.

A user-specified finite-time consensus is developed in the present work for a multiagent system to ensure consensus at a

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prespecified time instant. The interaction among agents (e.g., communication and information exchange) is modeled as a time-varying graph, where each edge is associated with a time-varying weight representing the time-varying interaction between neighboring agents. Consensus over such time-varying graph is then proven based on a time transformation and is guaranteed to be completed within a prespecified time frame. When considering time-varying graphs, the development of finite-time consensus can be more challenging. The stability analysis tools such as examining the eigenvalues of the Laplacian matrix of linear time-invariant systems in Refs. [22–25] are not applicable, since negative eigenvalues of linear time-varying systems does not indicate stability as discussed in Ref. [26, Chap. 4.6]. In addition, the consensus results over linear time-invariant graphs developed in Refs. [22–25] can be considered as a particular case of our work. In the companion paper [27], the current result is then generalized to dynamically perform cooperative engagement in the presence of unknown but bounded velocity of a target. However, different from Ref. [27] where the interagent interaction is modeled as a time-invariant graph, the current result is applicable to time-varying weighted graphs. To demonstrate the developed framework, finite-time rendezvous of a multiagent system is considered as an example application in this work, where agents are assumed to have limited communication capabilities (i.e., two agents can only communicate and exchange information within a certain distance). The agents are tasked to arrive at a common destination at a prespecified time instant, while preserving global network connectivity during rendezvous. Representative results on preservation of network connectivity include Refs. [28–31]. However, few existing results ensure rendezvous within a prespecified time frame while preserving network connectivity. A numerical simulation is provided to demonstrate the developed result. It is worth pointing out that the developed user-specified finite-time consensus is not limited to rendezvous, and it can be easily extended for applications such as flocking, formation control, and other collective tasks.

2 Preliminaries

2.1 Graph Theory. When a network of N agents is tasked to cooperatively perform collective tasks, the interagent interaction (e.g., information exchange) is generally modeled as a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where the set of vertices $\mathcal{V} = \{1, \ldots, N\}$ represents the agents, and the set of edges $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ denotes the interaction between agents. The edge $(i, j) \in \mathcal{E}$ can be either directed or undirected. An undirected edge (i, j) indicates that agent $i, j \in \mathcal{V}$ are able to exchange information with each other, while a directed edge (i, j) indicates that only agent j can receive information from agent i, but not vice versa. Each edge (j, i) is associated with a time-varying weight $a_{ij}(t)$ representing how agent i evaluates the information received from agent j at t. It is assumed that $0 \leq a_{ij}(t) \leq a_{\max}$ for any $i, j \in \mathcal{V}$, where $a_{\max} \in \mathbb{R}^+$ is a finite-positive constant. The positive weight $a_{ij}(t) > 0$ indicates that there exists an edge (j, i) in $\mathcal{E}(t)$, and $a_{ij} = 0$ otherwise. Since $a_{ij}(t)$ can vary between 0 and $a_{\max}$, the time-varying weights $\{a_{ij}(t)\}$ can reflect not only the time-varying interagent interaction but also instantaneous link creation (i.e., $a_{ij}(t) > 0$ with $a_{ij}(t^-) = 0$) or link failure (i.e., $a_{ij}(t) = 0$ with $a_{ij}(t^-) > 0$) in $\mathcal{G}(t)$.

The set of neighboring agents from which agent i can receive information from is defined as $\mathcal{N}_i(t) \triangleq \{j \in \mathcal{V} | (j, i) \in \mathcal{E}(t)\}$. The adjacency matrix is defined as $A(t) \triangleq [a_{ij}(t)] \in \mathbb{R}^{N \times N}$ and the Laplacian matrix $\mathcal{L}(t) \in \mathbb{R}^{N \times N}$ of graph $\mathcal{G}$ is defined as $\mathcal{L}(t) \triangleq D(t) - A(t)$, where $D(t) \triangleq \operatorname{diag}\{d_1, \ldots, d_N\}$ is a diagonal matrix with $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}(t)$ for $\forall i \in \mathcal{V}$. An undirected graph is connected if there exists a path connecting any two nodes. An out-tree is a directed graph where every node has only one parent except for one node, called the root, which has no parent. An oriented spanning tree is a directed subgraph that contains every node of $\mathcal{G}$.

2.2 Consensus Model. Consider a class of linear time-varying systems

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

where $x(t) = [x_1(t), \ldots, x_m(t)]^{\mathrm{T}} \in \mathbb{R}^m$ denotes an m-dimensional system state, and $A(t) \in \mathbb{R}^{m \times m}$ is a state transition matrix. The consensus, i.e., $x_1 = \cdots = x_m$ as $t \to \infty$, is established based on the following result.

Lemma 1. *Consider the linear time-varying system in Eq. (1) and a Lyapunov function $V(x) = \max\{x_1, \ldots, x_m\} - \min\{x_1, \ldots, x_m\}$ [32]. If the time-varying matrix $A(t)$ is a piecewise continuous function of time with bounded elements, $A(t)$ is a Metzler matrix[3] with zero row sums, and the time-varying graph corresponding to $A(t)$ is connected when an undirected graph is considered or has a spanning tree when a directed graph is considered, then $\dot{V} \leq 0$ for all $t \geq 0$ and consensus is achieved exponentially fast, i.e., $x_1(t) = \cdots = x_m(t)$ as $t \to \infty$.*

Most consensus results developed in the literature can be considered as a straightforward outcome of Lemma 1. For instance, consider a network of N agents whose interaction graph is modeled by the time-varying undirected graph $\mathcal{G}(t)$ described in Sec. 2.1. Classical consensus protocol indicates that each agent i updates its states $x_i \in \mathbb{R}$ according to $\dot{x}_i(t) = -\sum_{j \in \mathcal{N}_i} a_{ij}(t)(x_i(t) - x_j(t))$, which can be written in a compact form as $\dot{X}(t) = -\mathcal{L}(t)X(t)$, where $X(t) = [x_1(t), \ldots, x_N(t)]^{\mathrm{T}}$, and $\mathcal{L}(t)$ is the Laplacian matrix associated with the graph $\mathcal{G}(t)$. Since $-\mathcal{L}(t)$ is a *Metzler matrix with zero sums, if provided that* $\mathcal{G}(t)$ is connected, the states are ensured to achieve consensus, i.e., $x_1 = \cdots = x_N$ as $t \to \infty$, based on Lemma 1.

3 User-Defined Finite-Time Consensus

Motivated to achieve consensus within a desired time frame, the exponential consensus for networked systems in Lemma 1 is generalized in this section to a user-specified finite-time consensus:

Definition 1. (*User-Specified Finite-Time Consensus*) *Consider the networked system in Eq. (1) and a desired convergence time $t_f \in (0, +\infty)$. The networked system is said to achieve the user-specified finite-time consensus, if the consensus is achieved within the desired time frame t_f, i.e., $x_1(t) = \cdots = x_m(t)$ as $t \to t_f$.*

To facilitate the convergence analysis of the finite-time consensus, time transformation will be used in the subsequent development.

Lemma 2. *Let $\vartheta(t)$ denote a solution of the differential equation $\dot{x} = f(t, x)$ with the initial value $x_0(t_0)$* [34]. *The time transformation $t = \lambda(s)$ with a strictly increasing continuously differentiable function λ and the definition $\psi(s) = \vartheta(t)$ lead to*

$$\psi'(s) = \lambda'(s) f(\lambda(s), \psi(s))$$

with $\psi(\lambda^{-1}(t_0)) = x_0$, where $\psi'(s) = (d\psi(s)/ds)$ and $\lambda'(s) = (d\lambda(s)/ds)$.

Theorem 1. Consider a variant of the linear time-varying system in Eq. (1) as

$$\dot{x}(t) = u_i(t) \tag{2}$$

where the control input $u_i(t) = (c/(t_f - t))A(t)x(t)$ and $c \in \mathbb{R}^+$ is a constant control gain, and $t_f \in \mathbb{R}^+$ is the user-specified consensus time. If $A(t)$ is a piecewise continuous function of time with bounded elements, $A(t)$ is a Metzler matrix with zero row sums,

[3]Matrices with positive or zero off-diagonal entries are generally referred to as Metzler matrices [33].

and the time-varying graph corresponding to $A(t)$ is uniformly connected, the consensus is achieved within a desired finite time, i.e., $x_1(t) = \cdots = x_m(t)$ as $t \rightarrow t_f$.

Proof. To prove the finite-time consensus, i.e., $x_1(t) = \cdots = x_m(t)$ as $t \rightarrow t_f$, a time transformation based on Lemma 2 is applied to facilitate the convergence analysis. Since the time t is mapped onto $[0, t_f]$, the time transformation is designed as $\lambda(s) = t_f(1 - e^{-s})$ with $s \in [0, +\infty)$, where $\lambda(s)$ is a strictly increasing and continuously differentiable function with $\lambda(0) = 0$ and $\lim_{s\rightarrow\infty} \lambda(s) = t_f$. Applying the time transformation $t = \lambda(s) = t_f(1 - e^{-s})$ yields

$$\frac{dx(\lambda(s))}{d\lambda} = \frac{c}{t_f - \lambda(s)} A(\lambda(s))x(\lambda(s))$$

Using the fact that $(dx(\lambda(s))/ds) = (dx(\lambda(s))/d\lambda)(d\lambda/ds)$

$$\frac{dx(\lambda(s))}{ds} = \frac{d\lambda}{ds}\frac{c}{t_f - \lambda(s)} A(\lambda(s))x(\lambda(s)) \tag{3}$$

Define $\psi(s) \triangleq x(\lambda(s))$ and $\chi(s) \triangleq A(\lambda(s))$. Equation (3) can be simplified as

$$\psi'(s) = \lambda'(s)\frac{c}{t_f - \lambda(s)}\chi(s)\psi(s)$$

where $\psi'(s) = (d\psi(s)/ds)$ and $\lambda'(s) = (d\lambda(s)/ds)$. Using $\lambda'(s) = t_f e^{-s}$ yields

$$\begin{aligned}\psi'(s) &= t_f e^{-s}\frac{c}{t_f - t_f(1 - e^{-s})}\chi(s)\psi(s) \\ &= c\chi(s)\psi(s)\end{aligned} \tag{4}$$

Since $\lambda(s)$ is a strictly monotonically increasing function, $\chi(s)$ preserves the properties of $A(t)$ described in Lemma 1, i.e., $\chi(s)$ is a piecewise continuous function of s with bounded elements, Metzler matrix with zero sums and the time-varying graph corresponding to $\chi(s)$ is uniformly connected. Based on Lemma 1 and the fact that c is a positive constant, it concludes from Eq. (4) that $\psi_1(s) = \cdots = \psi_m(s)$ as $s \rightarrow +\infty$, which is equivalent to $x_1(t) = \cdots = x_m(t)$ as $t \rightarrow t_f$. □

Remark 1. Using the fact that

$$\frac{d\psi(s)}{ds} = \frac{dx(\lambda(s))}{ds} = \frac{dx(\lambda(s))}{d\lambda}\frac{d\lambda}{ds}$$

and the definition of $(dx(\lambda(s))/d\lambda) = (dx(t)/dt)$ and $(d\lambda/ds) = (t_f/e^s)$ yields

$$\psi'(s) = \frac{dx(t)}{dt}\frac{t_f}{e^s}$$

Based on Lemma 1, $\psi'(s) \rightarrow 0$ as $s \rightarrow +\infty$ from Eq. (4), since $\chi(s)$ is a Metzler matrix with zero sums. The fact that $\psi'(s) \rightarrow 0$ and $(d\lambda/ds) \rightarrow 0$ as $s \rightarrow +\infty$ indicates that $(dx(t)/dt)$ is bounded, since an unbounded $(dx(t)/dt)$ will lead to $0 \times \infty$ (i.e., the product of 0 and ∞), which is undefined and contradicts $\psi'(s) \rightarrow 0$ as $s \rightarrow \infty$. Therefore, $\dot{x}(t)$ in the system (2) is bounded as $t \rightarrow t_f$.

Remark 1 shows that the control input $u_i(t)$ in the system (2) is bounded, although the term $(c/(t_f - t))$ in Eq. (2) seems problematic intuitively as $t \rightarrow t_f$. Note that the numerator $cA(t)x(t)$ and the denominator $t_f - t$ in Eq. (2) both go to zero as $t \rightarrow t_f$. Since $cA(t)x(t)$ exponentially decreases to zero as shown in Theorem 1 while $t_f - t$ only linearly decreases to zero, based on L'Hospital's rule, it is expected that $u_i(t)$ is still bounded as $t \rightarrow t_f$. Nevertheless, more control actuation is required, if a small t_f is assigned. However, as long as $t_f > 0$ holds strictly, the control actuation will be bounded as indicated by Remark 1. Future research will focus on developing a saturated controller to achieve consensus within a prespecified finite-time frame.

4 Finite-Time Rendezvous With Connectivity Maintenance

To demonstrate the wide applications of the developed user-specified finite-time consensus algorithm, an example application of finite-time rendezvous for a multiagent system is considered where the agents are required to meet at a common location within a preset time frame.

4.1 Problem Formulation. A multiagent system composed of N agents is tasked to achieve rendezvous within a desired time t_f in a workspace $\mathcal{F}$, where $\mathcal{F}$ is a bounded disk area with radius R_w. Each agent is assumed to move with the single-integrator kinematics

$$\dot{p}_i(t) = u_i(t), \quad i = 1, \ldots, N \tag{5}$$

where $p_i(t) \triangleq [\,x_i(t) \quad y_i(t)\,]^{\mathrm{T}} \in \mathbb{R}^2$ denotes the position of agent i with respect to an inertial reference frame in $\mathcal{F}$, and $u_i(t) \in \mathbb{R}^2$ is the control input that represents the linear velocity of agent i.

It is assumed that each agent has a limited communication capability encoded by a disk area with radius R, which implies that two agents can only exchange information within a distance of R. The interagent interaction is modeled as an undirected graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ described in Sec. 2.1, where $\mathcal{V}$ represents the group of agents, and the time-varying edge set $\mathcal{E}(t) = \{(i,j) \in \mathcal{V} \times \mathcal{V} | d_{ij} \leq R\}$ indicates that the links between agents are established only when their relative distance $d_{ij} \triangleq \|p_i - p_j\| \in \mathbb{R}^+$ is less than R. Let $\mathcal{N}_i(t) = \{j \in \mathcal{V} | (i,j) \in \mathcal{E}(t)\}$ denote the neighbors of node i, which is a time-varying set since nodes may enter or leave the communication region of node i at any time instant. Due to the limited communication capabilities, agents have to stay close as a connected graph so that the agents can exchange information and coordinate their motion to perform rendezvous. To ensure connectivity, an escape region for each agent i is defined as the outer ring of the communication area with radius r, $R - \delta < r < R$, where δ is a predetermined buffer distance. Edges formed with any node $j \in \mathcal{N}_i$ in the escape region are in the danger of breaking. In contrast to most existing results in rendezvous problems that do not consider accomplishing the task within a desired time frame, the objective for the multiagent system is to converge to a common location within the given time frame t_f, while preserving global network connectivity using local information only (i.e., positions of one-hop neighbors). To achieve these goals, we assume that the initial graph $\mathcal{G}(0)$ is connected.

4.2 Control Design. Based on the artificial potential field-based approach (cf. Refs. [7] and [35]), a decentralized potential function $\varphi_i : \mathbb{R}^{2N} \rightarrow [0, 1]\ \forall i \in \mathcal{V}$ is developed for rendezvous as

$$\varphi_i = \frac{\gamma_i}{(\gamma_i^\alpha + \beta_i)^{1/\alpha}}, \quad i \in \mathcal{V} \tag{6}$$

where $\alpha \in \mathbb{R}^+$ is a tuning parameter,[4] $\gamma_i : \mathbb{R}^{2N} \rightarrow \mathbb{R}^+$ and $\beta_i : \mathbb{R}^{2N} \rightarrow \mathbb{R}^+$ are the goal function and the constraint function, respectively, that only use local position feedback from the neighboring agents.

[4]The control design in Eq. (5) is based on the framework of navigation functions, which ensures global convergence to the control objective of achieving rendezvous. It has been proven in our earlier paper [7] that an appropriate selection of the gain α ensures φ_i in Eq. (6) to be a qualified navigation function. Hence, the control gain α should be selected following the gain condition developed in Ref. [7].

The goal function in Eq. (6) is designed as

$$\gamma_i = \sum_{j\in\mathcal{N}_i} \frac{1}{2}\|p_i - p_j\|^2 \tag{7}$$

which is minimized when agent i and its neighbors $j \in \mathcal{N}_i$ converge to a common position. To preserve network connectivity, the constraint function $\beta_i : \mathbb{R}^{2N} \to [0, 1]$ in Eq. (6) is designed as

$$\beta_i = \prod_{j\in\mathcal{N}_i} b_{ij} \tag{8}$$

by only accounting for neighboring nodes. Particularly, $b_{ij}(p_i, p_j): \mathbb{R}^2 \to [0, 1]$ in Eq. (8) is a continuously differentiable function, designed as

$$b_{ij} = \begin{cases} 1, & s_{ij} \le (R-\delta)^2 \\ As_{ij}^3 + Bs_{ij}^2 + Cs_{ij} + D, & (R-\delta)^2 < s_{ij} < R^2 \\ 0, & s_{ij} \ge R^2 \end{cases} \tag{9}$$

and the partial derivative of b_{ij} with respect to s_{ij} is given by

$$\frac{\partial b_{ij}}{\partial s_{ij}} = \begin{cases} 0, & s_{ij} \le (R-\delta)^2 \\ 3As_{ij}^2 + 2Bs_{ij} + C, & (R-\delta)^2 < s_{ij} < R^2 \\ 0, & s_{ij} \ge R^2 \end{cases} \tag{10}$$

where $A = (-2/((R-\delta)^2 - R^2)^3)$, $B = (3((R-\delta)^2 + R^2)/((R-\delta)^2 - R^2)^3)$, $C = (-6(R-\delta)^2R^2/((R-\delta)^2 - R^2)^3)$, $D = (R^4(3(R-\delta)^2 - R^2)/((R-\delta)^2 - R^2)^3)$ and $s_{ij} \triangleq \|p_i - p_j\|^2$ denotes the square of the relative distance between node i and j. The b_{ij} in Eq. (9) is designed to preserve connectivity of node i and its neighbors $j \in \mathcal{N}_i$, which achieves its minimum value of zero as $s_{ij} = R^2$ and the maximum value of one when $s_{ij} \le (R-\delta)^2$. It is worth pointing out that, for each pair of nodes $(i, j) \in \mathcal{E}(t)$, b_{ij} is activated only when node j enters the escape region of node i and has a potential to break the exiting edge. When agent j moves in the area that $d_{ij} \le R - \delta$, b_{ij} imposes no constraint on its motion, since $(\partial b_{ij}/\partial s_{ij}) = 0$ as shown in Eq. (10).

Since γ_i and β_i in Eq. (6) are guaranteed to not be zero simultaneously from their definitions, the potential function φ_i in Eq. (6) achieves its minimum of zero when $\gamma_i = 0$ and its maximum of one when $\beta_i = 0$. Based on the potential function in Eq. (6), the controller of agent i is designed as

$$u_i(t) = -\frac{k_i}{t_f - t}\nabla_i\varphi_i \tag{11}$$

where $k_i \in \mathbb{R}^+$ denotes a positive control gain for agent i and $\nabla_i\varphi_i = \left[\frac{\partial\varphi_i}{\partial x_i} \quad \frac{\partial\varphi_i}{\partial y_i}\right]^{\mathrm{T}}$ denotes the gradient of φ_i with respect to p_i.

Lemma 3. *The controller designed in Eq. (11) ensures that the connectivity of the communication graph $\mathcal{G}$ is preserved when performing rendezvous.*

Proof. The initial graph $\mathcal{G}(0)$ is assumed to be connected. If every existing edge in $\mathcal{G}(t)$ is preserved for $t \ge 0$, the global connectivity will also be preserved. Consider a state p_i^* for agent i, where the relative distance between agent i and $j \in \mathcal{N}_i(t)$ satisfies $d_{ij}(p_i^*, p_j) = R$, which leads to $b_{ij}(p_i^*, p_j) = 0$ from Eq. (9) and indicates that the associated edge (i, j) is about to break. From Eq. (8), $\beta_i = 0$ when $b_{ij} = 0$, and the navigation function φ_i achieves its maximum value from Eq. (6). Since φ_i is maximized at p_i^*, no open set of initial conditions can be attracted to p_i^* under the negative gradient control law designed in Eq. (11). Therefore, the relative distance between agent i and j is maintained less than δ by Eq. (11), and the associated edge is also maintained. Repeating this argument for all pairs, every edge in $\mathcal{G}$ is maintained. □

Theorem 2. *Given a connected graph $\mathcal{G}(t)$, the controller designed in Eq. (11) ensures rendezvous within the preset time frame t_f, i.e., $p_1(t) = \cdots = p_N(t)$ as $t \to t_f$.*

Proof. From the designed potential function in Eq. (6)

$$\nabla_i\varphi_i = \frac{\alpha\beta_i\nabla_i\gamma_i - \gamma_i\nabla_i\beta_i}{\alpha(\gamma_i^\alpha + \beta_i)^{\frac{1}{\alpha}+1}} \tag{12}$$

where $\nabla_i\gamma_i$ and $\nabla_i\beta_i$ are bounded in the workspace $\mathcal{F}$ from Eqs. (7) and (8). Using Eqs. (7) and (8), the terms $\nabla_i\gamma_i$ and $\nabla_i\beta_i$ are determined as

$$\nabla_i\gamma_i = \sum_{j\in\mathcal{N}_i}(p_i - p_j) \tag{13}$$

and

$$\nabla_i\beta_i = \sum_{j\in\mathcal{N}_i} 2\left(\frac{\partial b_{ij}}{\partial s_{ij}}\right)\bar{b}_{ij}(p_i - p_j) \tag{14}$$

respectively, where $\bar{b}_{ij} \triangleq \prod_{l\in\mathcal{N}_i, l\ne j} b_{il}$.

Substituting Eqs. (13) and (14) into Eq. (12), $\nabla_i\varphi_i$ is rewritten as

$$\nabla_i\varphi_i = \sum_{j\in\mathcal{N}_i} m_{ij}(p_i - p_j) \tag{15}$$

where

$$m_{ij} = \frac{\alpha\beta_i - 2\left(\frac{\partial b_{ij}}{\partial s_{ij}}\right)\bar{b}_{ij}\gamma_i}{\alpha(\gamma_i^\alpha + \beta_i)^{\frac{1}{\alpha}+1}} \tag{16}$$

is non-negative and bounded, based on the definitions of γ_i, β_i, α, $\bar{b}_{ij}$, and the fact that $(\partial b_{ij}/\partial s_{ij}) \le 0$ from its definition in Eq. (10). Using the controller designed in Eqs. (11) and (15) yields the closed-loop system for each node i as

$$\dot{p}_i(t) = -\frac{k_i}{t_f - t}\sum_{j\in\mathcal{N}_i} m_{ij}(p_i - p_j)$$

which can be rewritten in a compact form as

$$\dot{\mathbf{p}}(t) = \frac{k_i}{t_f - t}(\pi(t) \otimes I_2)\mathbf{p}(t) \tag{17}$$

where $\mathbf{p}(t) = [p_1^{\mathrm{T}}, \ \ldots, \ p_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2N}$ denotes the stacked vector of p_i, I_2 is a 2×2 identity matrix, and the elements of $\pi(t) \in \mathbb{R}^{N\times N}$ are defined as

$$\pi_{ik}(t) = \begin{cases} -\sum_{j\in\mathcal{N}_i} m_{ij}, & i = k \\ m_{ik}, & k \in \mathcal{N}_i, i \ne k \\ 0, & k \notin \mathcal{N}_i, i \ne k \end{cases} \tag{18}$$

Since m_{ij} is non-negative from Eq. (16), the off-diagonal elements of $\pi(t)$ are positive or zero and its row sums are zero. Hence, $\pi(t)$ is a Metzler matrix with zero row sums. Since the communication graph $\mathcal{G}(t)$ is proven connected in Lemma 3, it is clear from Theorem 1 that the consensus is achieved within the time frame t_f, i.e., $p_1(t) = \cdots = p_N(t)$ *as* $t \to t_f$, which indicates that the group of agents converge to a common setpoint within the time frame t_f. □

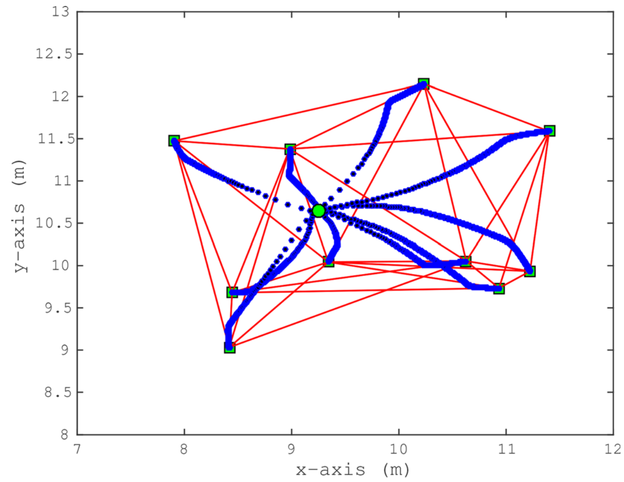

Fig. 1 The initial graph and trajectories of ten agents. The squares represent the initial positions of the agents and solid lines connecting agents indicate the interagent communication. The trajectory of each agent is represented by dots, which indicates that all agents converge to a common setpoint denoted by the circle.

Remark 2. Like most existing results in rendezvous, collision avoidance among agents is not considered in the present work, since it conflicts with the objective of meeting at a common setpoint. An alternative to include collision avoidance for rendezvous problems is to divide the workspace into two areas: a rendezvous area and a collision-free area. The rendezvous area could be a user-defined proximity area around the rendezvous point in which agents are required to perform rendezvous only, while the collision-free area is an area distant from the rendezvous point in which agents are navigated toward the rendezvous area and avoid collision with other agents. In our earlier work [7], a navigation function-based decentralized controller is developed for multiagent systems to perform collective tasks with ensured collision avoidance. The control strategy developed in Ref. [7] could be extended to navigate the agents toward the rendezvous point with ensured collision avoidance within the collision-free area. After entering the rendezvous area, the finite-time consensus developed in Eq. (11) can then be applied to complete rendezvous.

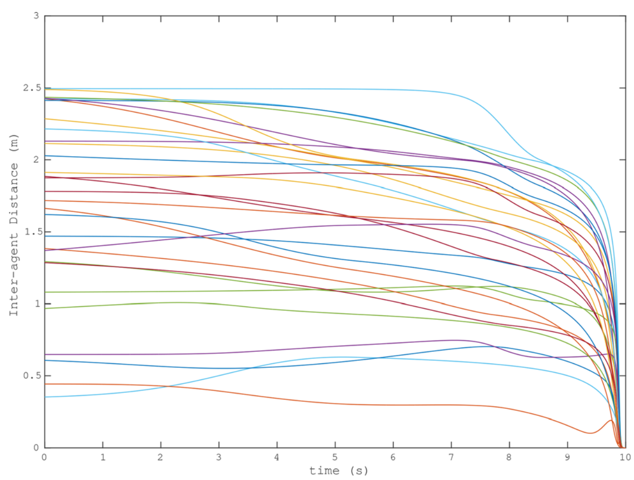

Fig. 2 The evolution of interagent distance, which is maintained less than the communication radius $R = 2.5$ m, indicating that every existing communication link is preserved when performing rendezvous. Since all edges converge to 0 at $t_f = 10$ s, the finite-time rendezvous is completed within the desired time frame.

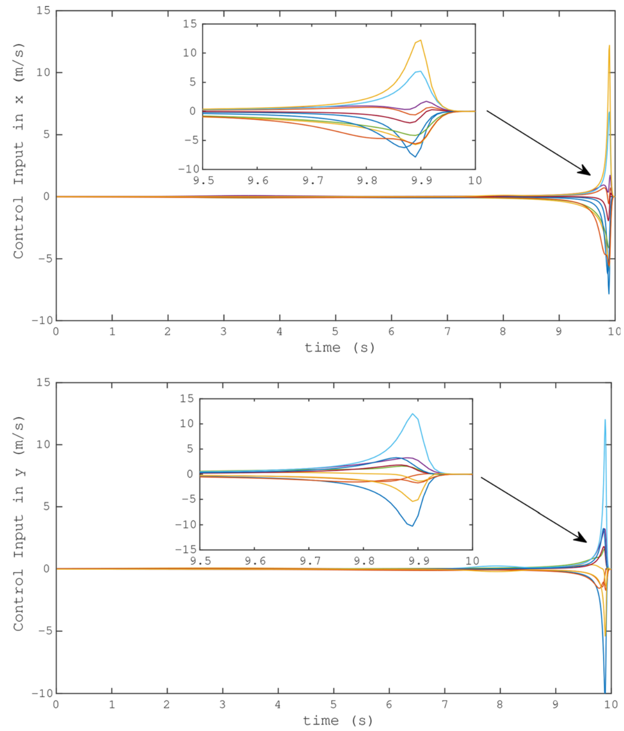

Fig. 3 The control input of each agent, which indicates that the control input is bounded as $t \to t_f$

Remark 3. Although rendezvous is considered as an example application in the present work, the developed finite-time consensus framework in Eq. (2) is not limited to rendezvous and is applicable to various collective tasks to ensure mission accomplishment within a desired time frame. For instance, it is well known that most formation control problems can be formulated as consensus problems. When modifying the objective function in Eq. (7) to encode formation control as a consensus problem, the current result can be straightforwardly extended to perform finite-time formation control. Since other applications such as flocking and synchronization can also be formulated as consensus problems, the developed framework in Eq. (2) is also applicable to such applications.

4.3 Simulation. To demonstrate the performance of the developed finite-time consensus in Eq. (11), numerical simulation is provided for a group of ten mobile agents with the kinematics in Eq. (5). The agents are required to meet at a common location with a preset time frame $t_f = 10$ s. The limited communication zone for each robot is assumed as $R = 2.5$ m and $\delta = 0.5$ m. The tuning parameter α in Eq. (6) is selected to be $\alpha = 1.1$. As shown in Fig. 1, the group of mobile agents is arbitrarily deployed in the workspace and forms a connected graph, where the squares denote the initial positions of agents, and the solid lines represent the interagent communication. The trajectory for each agent is shown by dots in Fig. 1, which indicates that the group of agents converge to a common setpoint. To demonstrate the preservation of network connectivity, the evolution of the interagent distance is plotted in Fig. 2. Note that the interagent distance is maintained less than the communication radius $R = 2.5$ m, which indicates that each existing link is maintained and the global connectivity of the underlying graph is preserved. Since all edges in Fig. 2 converge to zero at the time $t_f = 10$ s, the finite-time rendezvous is accomplished within a preset time frame. The control inputs are shown in Fig. 3, indicating that the controls are bounded as $t \to t_f$.

5 Conclusion

The finite-time consensus framework is developed in the current work to ensure consensus within a desired time frame over a time-varying graph. The effectiveness of the developed framework is demonstrated via a finite-time rendezvous problem, where agents with limited communication capabilities are desired to meet at a common location at a desired time instant, while preserving global network connectivity. Since only single-integrator kinematics is considered, future work will aim to extend the current results to agents with more general dynamics such as higher-order dynamics, Euler–Lagrange dynamics, or mobile agents with nonholonomic constraints. Additional research will also focus on improving the system performance by developing controllers robust to external disturbances and achieving rendezvous at a desired destination.

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